# AP Calculus BC

# Unit 2 – Introduction to Differentiation

**AP Calculus BC – Worksheet 7** **Average Rates of Change**

1 Find the value of *a* so that the function is continuous.

$$f(x)=\begin{cases} x^2+x+a, & x<1 \\ x^3, & x\geq 1 \end{cases}$$

2 Explain why the equation $e^{-x}=x$ has at least one solution.

3 Sketch one possible graph for a function $f(x)$ that has the stated properties:

$f(4)$ exists

$\lim\limits_{x\to 4} f(x)$ exists

$f(x)$ is not continuous at $x=4$.

**In exercises 4-7, find the average rate of change of the function over each interval.**

4 $f(x)=x^3+1$

a) $[2,3]$ b) $[-1.1]$

5 $f(x)=e^x$

(a) $[-2,0]$ b) $[1,3]$

6 $f(x)=\ln x$

(a) $[1,4]$ b) $(100,103)$

7 $f(x)=2+\cos x$

(a) $[0,\pi]$ b) $[-\pi,\pi]$

The table below shows selected values for a function, $f(x)$, at various values of *x*.

| $x$ | 0 | 1 | 2 | 5 | 9 |
|---|---|---|---|---|---|
| $f(x)$ | 14 | 18 | 24 | 32 | 44 |

8) Find the average rate of change of the function over the interval $[1,2]$.

9) Find the average rate of change of the function over the interval $[5,9]$.

10) Estimate the slope of the function when $x=3$.

11 A tank of water is draining so that the number of gallons of water in the tank after *t* minutes is determined by $Q(t)=10(4-t)^2$. Find the average rate of change at which the water has drained during the first two minutes.

12 Grass clippings are placed in a bin, where they decompose. For $0\leq t\leq 30$, the amount of grass clippings remaining in the bin is modeled by $A(t)=6.687(0.931)^t$, where $A(t)$ is measured in pounds and *t* is measured in days. Find the average rate of change of $A(t)$ over the interval $0\leq t\leq 30$. Indicate units of measure.

13 A ball dropped from a state of rest at time $t=0$ travels a distance $s(t)=4.9t^2$ m in *t* seconds.

a) How far does the ball travel during the time interval $[2,2.5]$?

b) Compute the average rate of change (velocity) of the ball over $[2,2.5]$.

c) Compute the average rate of change for the time intervals in the table.

| Interval | $[2,2.01]$ | $[2,2.001]$ | $[2,2.00001]$ |
|---|---|---|---|
| Average Rate of Change over interval | | | |

**AP Calculus BC – Worksheet 8** **The Definition of the Derivative/Differentiability**

In Exercises 1-3, use the definition

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

to find the derivative of the given function at the indicated point.

| | | |
|---|---|---|
| 1) $f(x)=\frac{1}{x},\ a=2$ | 2) $f(x)=x^2+4,\ a=1$ | 3) $f(x)=x^3+x,\ a=0$ |

In Exercises 4-6, use the alternate form of the definition

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

to find the derivative of the given function at the indicated point.

| | | |
|---|---|---|
| 4) $f(x)=x^2+4,\ a=1$ | 5) $f(x)=\sqrt{x+1},\ a=3$ | 6) $f(x)=2x+3,\ a=-1$ |

7) For the function $f(x)=\begin{cases}3x^2-4, & x<0\\ 3x-4, & x\geq 0\end{cases}$, determine if $f(x)$ is differentiable at $x=0$.

$$g(x)=\begin{cases}(x+1)^2, & x\leq 0\\ 2x+1, & 0<x<5\\ (6-x)^2, & x\geq 5\end{cases}$$

8) Given the function $g(x)$ above:

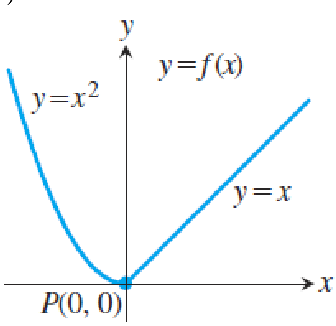
a) Determine if $g(x)$ is differentiable at $x=0$

b) Determine if $g(x)$ is differentiable at $x=5$

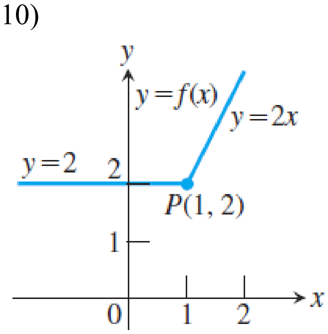
c) State the values of $x$ for which $g(x)$ is differentiable.

For 9 and 10, find all points where $f$ is not differentiable. Verify by comparing right-hand and left-hand derivatives.

9)




10)



**AP Calculus BC - Worksheet 9** **Rules for Derivatives**

Differentiate the following functions. Do not simplify the answer

| | | |
|---|---|---|
| 1) $g(t)=6t^{\frac{5}{3}}$ | 2) $B(x)=\dfrac{8x^2-6x+11}{x-1}$ | 3) $f(s)=15-s-4s^2-5s^4$ |
| 4) $G(v)=\dfrac{v^3-1}{v^3+1}$ | 5) $f(x)=3x^2+\sqrt[3]{x^4}$ | 6) $g(t)=\dfrac{\sqrt[3]{t^2}}{3t-5}$ |
| 7) $p(x)=1+\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}$ | 8) $k(x)=\left(2x^2-4x+1\right)\left(6x-5\right)$ | 9) $h(x)=x^{\frac{2}{3}}\left(3x^2-2x+5\right)$ |
| 10) $M(x)=\dfrac{2x^3-7x^2+4x+3}{x^2}$ | 11) $f(x)=\dfrac{4x-5}{3x+2}$ | 12) $f(x)=\dfrac{1}{1+x+x^2+x^3}$ |

| | |
|---|---|
| 13 | Sketch the graph of a continuous function *f* with $f(0)=-1$ and $f'(x)=\begin{cases}1, & x<-1\\ -2, & x>-1\end{cases}$. |
| 14 | **True or False** If $f(x)=x^2+x$, then $f'(x)$ exists for every real number *x*. Justify your answer. |
| 15 | Let $f(x)=4-3x$. Which of the following is equal to $f'(-1)$? <br> (B) 7 (C) -3 (D) 3 (E) does not exist |
| 16 | Find the unique value of *k* that makes the function, $f(x)=\begin{cases}x^3, & x\le 1\\ 3x+k, & x>1\end{cases}$ differentiable at $x=1$. |

**AP Calculus BC – Worksheet 10** **Tangents, Normals, and Higher Order Derivatives**

1. For what values of $x$ does the graph of $y = 2x^3 + 3x^2 - 12x + 1$ have a horizontal tangent?

2. For what values of $x$ does the graph of $f(x) = (x^2 + 1)(x + 3)$ have a horizontal tangent?

3. Find the equations of the tangent and normal lines to the curve $y = x^2(3 - x)$ when $x = -2$.

4. Find the equations of the tangent and normal lines to the curve $f(x) = \sqrt{x}$ when $x = 4$.

5. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

6. Find the equation of the line perpendicular to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$.

7. Find an equation of the tangent line to the curve $y = (x^3 - 3x + 1)(x + 2)$ when $x = 1$.

8. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is parallel to the $x$-axis.

9. Find $f''(x)$ for $f(x) = \frac{1}{x}$.

10. Find $\frac{d^2y}{dx^2}$ for $y = \sqrt{x}$.

Use the given table for problems 12-15.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 5 | $-1$ | 1 | 2 |
| 2 | 4 | $-1$ | 3 | $\frac{3}{2}$ |
| 3 | 3 | $-1$ | 4 | 1 |
| 4 | 2 | $-1$ | 5 | 1 |
| 5 | 1 | 0 | 6 | $-\frac{1}{2}$ |
| 6 | 2 | 1 | 4 | $-2$ |

11. Given $h(x) = f(x) + g(x)$, find $h'(2)$.

12. Given $d(x) = f(x) - g(x)$, find $d'(3)$

13. Given $p(x) = f(x) \cdot g(x)$, find $p'(4)$.

14. Given $q(x) = \frac{f(x)}{g(x)}$, find $q'(2)$.

15. Given $m(x) = f(g(x))$, find $m'(6)$.

**AP Calculus BC – Worksheet 11** **Derivative of Trig Functions**

Differentiate the following. Do not simplify your answer.

1. $f(x)=\sin x\cot x$

2. $f(x)=\dfrac{\tan x}{1+x^2}$

3. $g(w)=\dfrac{1+\sec w}{1-\sec w}$

4. $k(v)=\dfrac{\csc v}{\sec v}$

5. $k(x)=\sin x+2x^3+4\tan x$

6. $F(x)=\dfrac{\cos x}{1-\sin x}$

7. $r(a)=\csc(a^3)$

8. $H(s)=\cot(s^2-4\sqrt{s})$

9. $f(x)=5\tan\left(\cos x\right)$

10. $f(x)=\cos x+3x^2$

11. $p(w)=\tan\sqrt{w}$

12. $P(v)=\sin 3v\csc 3v$

13. $N(x)=\sin x-5\cos x$

14. $h(x)=x^3\csc x$

15. $L(x)=\tan x\sec x$

| | |
|---|---|
| 16 | Find the equations for the lines that are tangent and normal to the graph of $f\left(x\right)=\sin x+3$ at $x=\pi$. |
| 17 | Find the equation of the normal line to $f\left(x\right)=\sin x+\cos x$ at $x=\pi$. |
| 18 | Determine all values of $x$ in the interval $\left(0,2\pi\right)$ for which $f\left(x\right)=\cos 2x$ has horizontal tangents. |

**AP Calculus BC – Worksheet 12** **The Package Rule**

Differentiate the following. Do not simplify your answer.

1. $f(x)=\left(7x+\sqrt{x}\right)^{-8}$

2. $f(x)=x^3\left(2x-5\right)^5$

3. $g(w)=\csc^4\left(w^5-w^3\right)$

4. $k(v)=\sin^2\left(5\pi v-4\right)$

5. $k(x)=\sin^{-3}x-\cos^3 x$

6. $F(x)=\sqrt{-3-9x}$

7. $r(a)=\left(4a^3+5\right)^{\frac{3}{2}}$

8. $H(s)=\sqrt[3]{12s^2+8}$

9. $y=\dfrac{1}{\left(4x+3\right)^4}$

10. $f(x)=\left(\dfrac{x-3}{x-8}\right)^6$

11. $p(w)=\left(\csc w+\cot w\right)^{-1}$

12. $P(v)=\left(\dfrac{-\cos v}{1+\sin v}\right)^2$

**Answers (not simplified)**

| | |
|---|---|
| 1. $f'(x)=-8\left(7x+\sqrt{x}\right)^{-9}\left(7+\frac{1}{2}x^{-\frac{1}{2}}\right)$ | 2. $f(x)=5x^3\left(2x-5\right)^4\cdot 2+3x^2\left(2x-5\right)^5$ |
| 3. $g'(w)=4\csc^3\left(w^5-w^3\right)\left(-\csc\left(w^5-w^3\right)\cot\left(w^5-w^3\right)\right)\left(5w^4-3w^2\right)$ | 4. $k'(v)=2\sin\left(5\pi v-4\right)\cos\left(5\pi v-4\right)\cdot 5\pi$ |
| 5. $k'(x)=-3\sin^{-4}x\cos x-3\cos^2 x\left(-\sin x\right)$ | 6. $F'(x)=\frac{1}{2}\left(-3-9x\right)^{-\frac{1}{2}}\left(-9\right)$ |
| 7. $r'(a)=\frac{3}{2}\left(4a^3+5\right)^{\frac{1}{2}}\left(12a^2\right)$ | 8. $H'(s)=\frac{1}{3}\left(12s^2+8\right)^{-\frac{2}{3}}\left(24s\right)$ |
| 9. $y'=-4\left(4x+3\right)^{-3}\left(4\right)$ | 10. $f'(x)=6\left(\frac{x-3}{x-8}\right)^5\left(\frac{(x-8)-(x-3)}{(x-8)^2}\right)$ |
| 11. $p'(w)=-1\left(\csc w+\cot w\right)^{-2}\left(-\csc w\cot w-\csc^2 w\right)$ | |
| 12. $P'(v)=2\left(\frac{-\cos v}{1+\sin v}\right)\left(\frac{(1+\sin v)(-\sin v)-(-\cos v)(\cos v)}{(1+\sin v)^2}\right)$ | |

**AP Calculus BC – Worksheet 13** **Unit 2 - Intro to Differentiation Review**

| | |
|---|---|
| 1 | Use left- and right-hand derivatives to determine if $f(x)=\begin{cases} x^2, & x<1 \\ 3x-2, & x\geq 1 \end{cases}$ is differentiable at $x=1$ |
| 2 | Find $y'$ if $y=x^4\left(4\sqrt{x}-3\sqrt[3]{x}\right)$ |
| 3 | Given $f(x)=\dfrac{x^2+3x}{x-2}$, find $f'(x)$ . |
| 4 | Find $y''$ if $y=\left(x^2-2x+1\right)^3$ . |
| 5 | Write the equations of the tangent and normal lines to the graph of $f(x)=x^3-5x+2$ when $x=-2$ . |
| 6 | Find the point on the graph of $g(x)=x^2+3x+4$ where the tangent line is parallel to the line $4x-y=7$ . |
| 7 | Write the equations of the tangent and normal lines to $f(x)=\sqrt{x^2-2x}$ when $x=3$ . |

In questions 8-12, differentiable functions *f* and *g* have values shown in the table below.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 2 | 1 | 5 | -4 |
| 1 | 3 | 2 | 3 | -3 |
| 2 | 5 | 3 | 1 | -2 |
| 3 | 10 | 4 | 0 | -1 |

| | |
|---|---|
| 8 | If $A=f+2g$ , then $A'(3)=$ |
| 9 | If $B=f\cdot g$ , then $B'(2)=$ |
| 10 | If $C=\dfrac{g}{f}$ , then $C'(1)=$ |
| 11 | If $D=\dfrac{1}{g}$ , then $D'(0)=$ |
| 12 | If $E=\dfrac{f}{g}$ , then $E'(3)=$ |

| | |
|---|---|
| 13 | Find $y'$ if $y=x^4\left(4\cos x-3\tan x\right)$ |
| 14 | Given: $f(x)=\dfrac{\cos x}{1+\tan x}$ . Find $f'(x)$ |
| 15 | Find $y''$ if $y=x^7\left(3\csc x\right)$ |
| 16 | Differentiate $y=\dfrac{\tan x}{2x+\csc x}$ |
| 17 | Differentiate $y=\sin\left(3x-4\right)$ |
| 18 | Find $f'(x)$ for $f(x)=\tan^3\left(4x^6-2x\right)$ |
| 19 | Find $y'$ for $y=\left(x^3+2\right)^4\left(\cot x-2x\right)^5$ |

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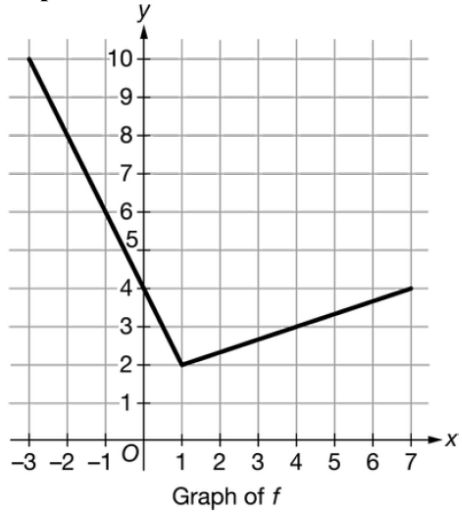
**A graphing calculator is required for this problem.**




1. Let *f* be the continuous function defined on $[-3,7]$ whose graph, consisting of two line segments, is shown above.

   Let *g* and *h* be the functions defined by $g(x)=\left(x^2+5x\right)^{\frac{1}{3}}$ and $h(x)=7\cos x+x^3$.

   a) The function *N* is defined by $N(x)=f(x)g(x)$. Find $N'(-1)$. Show the work that leads to your answer.

   b) The function *P* is defined by $P(x)=\dfrac{g(x)}{5f(x)}$. Find $P'(4)$. Show the work that leads to your answer.

   c) Find the value of *x* for $-3<x<1$ such that $f'(x)=h'(x)$.

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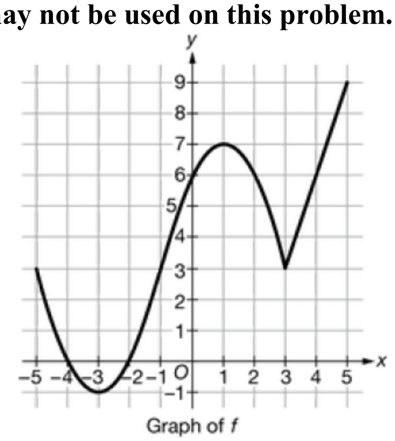
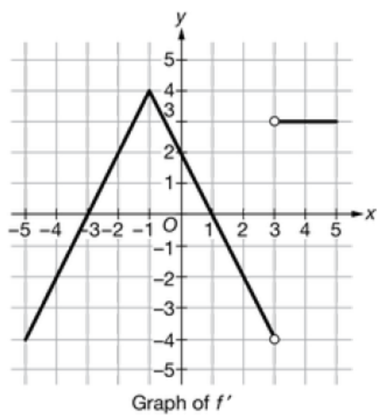
**A graphing calculator may not be used on this problem.**




2. The graphs of the function *f* and its derivative *f*′ are shown above for $-5\le x\le 5$.

   a) Find the average rate of change of *f* over the interval $-5\le x\le 5$. For how many values of *x* in the interval $-5\le x\le 5$ does the instantaneous rate of change of *f* equal the average rate of change of *f* over that interval?

   b) Write an equation for the line tangent to the graph of *f* at $x=2$.

   c) For each $\lim\limits_{x\to -1}\dfrac{f(x)-f(-1)}{x-(-1)}$ and $\lim\limits_{x\to 3}\dfrac{f(x)-f(3)}{x-3}$, find the value or give a reason why it does not exist.

   d) Let *g* be the function defined by $g(x)=f(x)\cdot\ln x$. Find $g'(4)$.

**A graphing calculator may not be used for this problem**

| ***t*** **(minutes)** | 0 | 15 | 45 | 70 | 90 | 100 |
|---|---|---|---|---|---|---|
| $A(t)$ **(automobiles)** | 0 | 30 | 190 | 250 | 405 | 600 |

3. Prior to a sporting event, the number of automobiles that have entered a stadium parking is modeled by the differentiable function *A*, where *t* is the number of minutes since the parking lot opened. Values of $A(t)$ for selected values of *t* are given in the table above.

   a) According to the model, what is the average rate at which automobiles enter the parking lot, in automobiles per minute, over the time interval $45 \le t \le 90$?

   b) Write $A'(95)$ as the limit of a difference quotient. Use the data in the table to approximate $A'(95)$. Show the computations that lead to your answer.

   c) What is the shortest time interval during which it is guaranteed that $A(t) = 400$ for some time *t* in the interval? Justify your answer.

   d) For $0 \le t \le 45$, the function *f* defined by $f(t) = 227t^2 + 89t$ models the number of automobiles that have entered the parking lot, where *t* is the number of minutes since the parking lot opened. Find $f'(10)$, the rate at which automobiles enter the parking lot in automobiles per minute, at time $t = 10$ minutes.